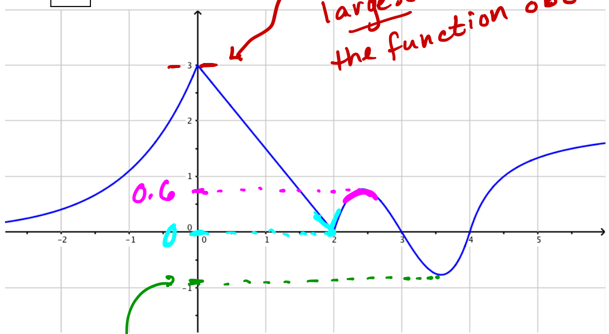
Lecture Notes: 1-4 Maximum and Minimum Values (part 1)

Motivating Examples

A

3 is the (absolute) maximum or largest value the function obtains

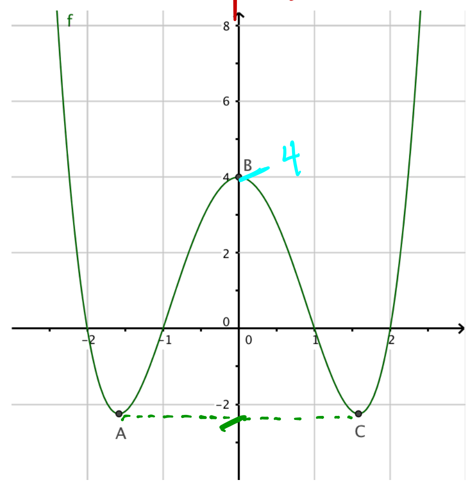


approximately −0.9 is the (absolute) minimum or the smallest value the function obtains

- ALL other y-values are between $y=-0.9$ and $y=3$
- There are other places that are "locally" smallest or largest y-values.

B

Hey! No maximum! y-values keep growing larger.



- There is an absolute minimum, approx. $y=-2.2$.
- This minimum occurs at two different places: $x \approx -1.5$ and $x \approx 1.5$.
- There is a local maximum of $y=4$ which occurs at $x=0$

Why would one care?
Wouldn't you like to maximize your profit?
or minimize your fuel consumption?

Definitions: Let $f(x)$ be a function with domain D and let c be an x-value in D. Then the y-value $f(c)$ is:

1. **an absolute maximum** if $f(c) \geq f(x)$ for all x in D (that is, $f(c)$ is the largest y-value possible)
2. **an absolute minimum** if $f(c) \leq f(x)$ for all x in D. (that is, $f(c)$ is the smallest possible y-value.)
3. **a local maximum** if $f(c) \geq f(x)$ for all x close to c.
4. **a local minimum** if $f(c) \leq f(x)$ for all x close to c

Do you see the difference?

Are we all on the same page?

1. What sort of *category* is a maximum (or minimum)? (Animal, vegetable, number, point, x-value, y-value, mineral...?)

- a max or min is a $\boxed{\text{NUMBER}}$. (not a point (x,y), or a function, ...)
- it is always a function value or output value or y-value.

2. Can function have more than ONE maximum (or minimum)? Yes and no (ha ha ha)

- A function can have at most one absolute maximum; at most one absolute minimum.
- The absolute max/min may occur at multiple x-values
- A function can have multiple different local maximums or minimums.

3. Can a function have neither a maximum nor a minimum?

Sure. Some examples include $f(x)=x^3$, $g(x)=e^x$, $h(x)=\frac{1}{x}$.

4. Looking at our earlier pictures, at what sort of places do maximums and minimums appear?

- smooth turn around points:

where $f'(x)=0$

- Sharp turn around points

where $f'(x)$ fails to exist.

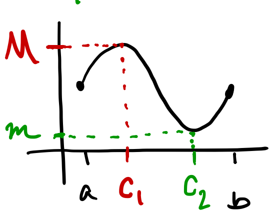
Powerful Tool #1: The Extreme Value Theorem

Given a function $f(x)$ such that

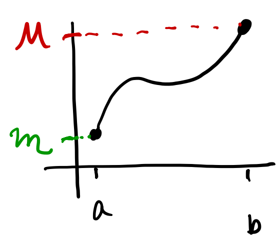
1. the domain is restricted to a closed interval $[a,b]$ and
2. $f(x)$ is continuous on $[a,b]$,

then $f(x)$ is *guaranteed* to have a maximum and a minimum on the interval $[a,b]$.

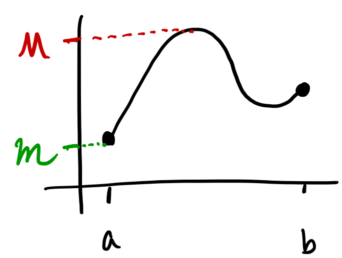
pictures



max (M) and min (m) occur at turnaround points ($x=c_1$ and $x=c_2$)



max (M) and min (m) occur at end points of interval

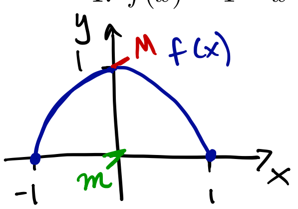


max (M) at turnaround.

min (m) at end point

PRACTICE PROBLEMS: For each function with designated region, sketch the graph to determine its absolute maximum and its absolute minimum, if they exist. If they do not exist, explain why the Extreme Value Theorem is not violated.

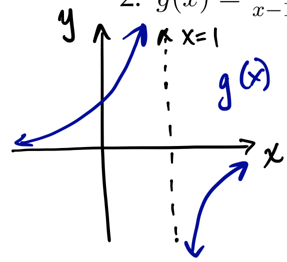
1. $f(x) = 1 - x^2$ on $[-1, 1]$



max: $y=1$ at $x=0$

min: $y=0$ at $x=-1, x=1$

2. $g(x) = \frac{-2}{x-1}$ on $[0, 2]$

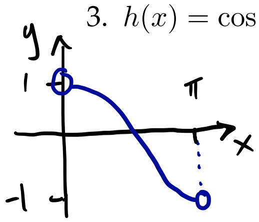


No maximum.

No minimum.

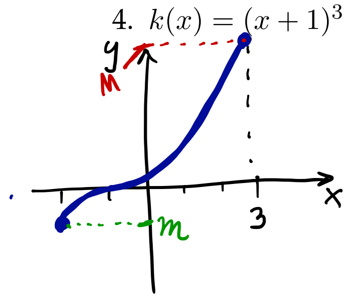
$g(x)$ is not continuous on the interval $[0,2]$.

3. $h(x) = \cos x$ on $(0, \pi)$



No max. No min. The interval isn't closed.

4. $k(x) = (x+1)^3$ on $[-2, 3]$



$f(-2) = (-1)^3 = -1$

$f(3) = 4^3 = 64$

max: $y=64$ at $x=3$

min: $y=-1$ at $x=-2$

POWERFUL TOOL #2: Critical Points

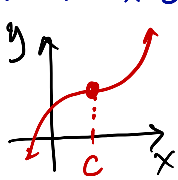
Definition: A **critical number** of a function $f(x)$ is an x-value c in the domain of $f(x)$ such that either

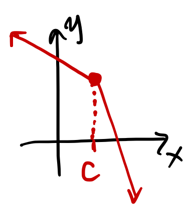
(a) $f'(c) = 0$ or (b) $f'(c)$ is undefined.

Why do we care about critical points?

- Critical numbers give x-values where we may find max's and min's.
- Note: we must check end points, too, if there are any.
- Just because $x=c$ is a critical point does not mean $f(c)$ is necessarily a max or min.

Example:





Critical points NOT corresponding to max's or mins.

- In Summary

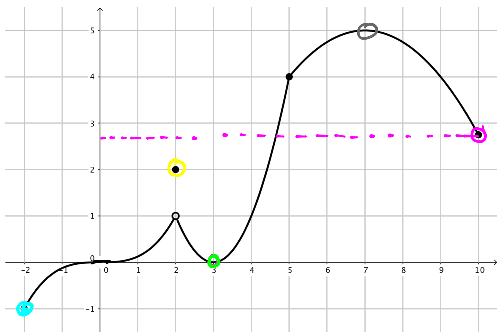
max's and min's occur at "turn around points" and end points

or

critical points and end points

More Practice Problems: For each function below, (a) find all critical points (b) identify all local maximums and local minimums (if any) and *where they occur* (c) identify all absolute maximum and minimums and *where they occur*. If no domain is explicitly stated, assume you are using the natural domain of the function as written. **You are expected to provide clear coherent explanations of how you deduced your answers.**

1. $f(x)$ is graphed below:



(a) $x = 0, 2, 3, 5, 7$

$f' = 0$

f' undefined

(b)

local mins	-1	0	2.8
location	-2	3	10

← y-values
← x-values

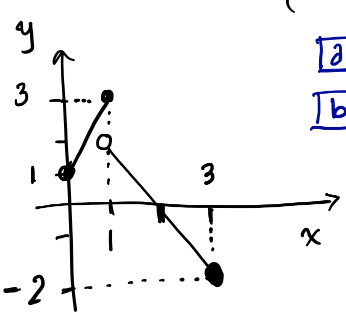
local maxs	2	5
location	2	7

← y-values
← x-values

(c) abs min: $y = -1$ at $x = -2$

abs max: $y = 5$ at $x = 7$

2. $f(x) = \begin{cases} 2x+1 & \text{if} \quad 0 \le x < 1 \\ 4 - 2x & \text{if} \quad 1 \le x \le 3 \end{cases}$



a. critical pts: $x = 1$

b.

local mins:	1	-2
location	0	3

← y's
← x's

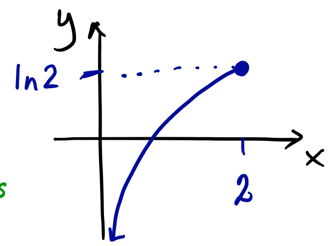
local maxs:	3
location	1

← y's
← x's

c. absolute max: 3 at $x=1$

absolute min: -2 at $x=3$

3. $g(x) = \ln x$ on $(0, 2]$



a. critical pts: none

b. local mins: none

local max: $y = \ln 2$

location: $x = 2$

c. absolute max: $\ln 2$ at $x = 2$

absolute min: none

4. $f(t) = t^4 + t^3 + t^2 + 1$

$f'(t) = 4t^3 + 3t^2 + 2t$

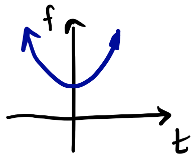
$= t(4t^2+3t+2) = 0$ no roots

So $f'(t)=0$ if $t=0$.

a) critical pts: $t=0$

end pts: none

using the graph:



b) local min: $y=1$ at $t=0$

local max's: none

c) absolute mins: $y=1$ at $x=0$

absolute max: none

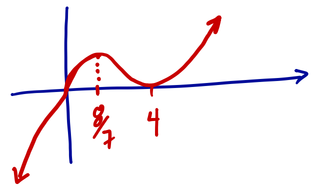
5. $h(x) = x^{4/5}(x-4)^2$

factor out

$h'(x) = \frac{4}{5}x^{-1/5}(x-4)^2 + x^{4/5} \cdot 2(x-4)' = \frac{2(x-4)}{x^{1/5}}\left[\frac{2}{5}(x-4)+x\right] = \frac{2(x-4)(7x-8)}{5x^{1/5}}$

a. critical numbers: $x = 4, 8/7, 0$

Graph Sketch



b. local min: $y=0$
location: $x=4$

local max: $h(8/7) = (\frac{8}{7})^{4/5}(\frac{8}{7}-4)^2 \approx 8.81$
location: $x = 8/7$

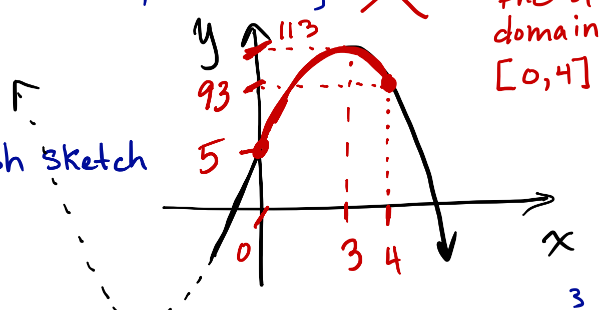
c. absolute min: none
absolute max: none

6. $f(x) = 5 + 54x - 2x^3$ on $[0, 4]$

$f'(x) = 54 - 6x^2 = 6(3-x)(3+x)$

a. critical pts: $x=3$, $\cancel{x=-3}$ not in the specified domain $[0,4]$

graph sketch



b. local max: $y = f(3) = 5 + 54 \cdot 3 - 2 \cdot 3^3 = 113$

location: $x=3$

local mins	5	$f(4)=93$
location $x=$	0	4

c. absolute max: $y=113$ at $x=3$.

absolute min: $y=5$ at $x=0$.

7. $g(x) = x + \frac{1}{x}$ on $[0.2, 4]$

$g'(x) = 1 + (-1)x^{-2} = 1 - \frac{1}{x^2}$

a. critical pts: $\cancel{x=0}$ ← oops! Not in domain., $x=1$ (not $x=-1$)

Skip graph. Just check y-values

end points

critical pt.

x	0.2	4	1
y	5.2	4.25	2

b. local min: $y=2$ at $x=1$

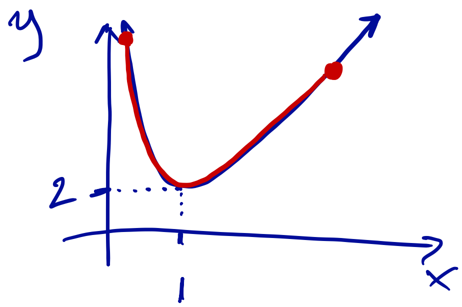
local max's $y=$	5.2	4.25
location $x=$	0.2	4

c. absolute max $y=5.2$ at $x=0.2$

absolute min $y=2$ at $x=1$

* OK. Can't help myself!

Graph $f(x) = x + \frac{1}{x}$ on $[0, \infty)$



$[0.2, 4]$

Note to Self:

If time permits, state explicity "Closed Interval Algorithm."
What it does and does NOT tell you.